Counting

Lecture 9   Mar 07, 2021

Coefficients of polynomial expansions

Polynomial expansion.

- $(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2 = 1\,x^2 + 2\,xy + 1\,y^2$ 😀
- $(x+y)^3 = (x+y)^2(x+y) = (x^2 + 2xy + y^2)(x+y) = 1\,x^3 + 3\,x^2y + 3\,xy^2 + 1\,y$^3

What is the expansion formula for arbitrary $(x+y)^n$?

n choose k: $\binom{n}{k}$

Expansion formula:

$$(x+y)^n = (x+y)\cdots(x+y) = \binom{n}{0}\ x^n + \binom{n}{1}\ x^{n-1}y + \cdots + \binom{n}{k}\ x^{n-k}y^k + \cdots + \binom{n}{n}\ y^{\mathrm{n}}$$

- **What is $\binom{n}{k}$**: The number of ways you can choose k things from a set of size n

- **Why the coefficients are $\binom{n}{k}$**:

To get $x^{n-k}y^k$ you need to choose y from k of the n terms in the product

Examples

- $(x+y)^2 = 1\,x^2 + 2\,xy + 1\,y^2$ 😀

So $\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

- $(x+y)^3 = 1\,x^3 + 3\,x^2y + 3\,xy^{\mathbf{2}} + 1\,y{}^\wedge 3$

So $\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$

Properties

$$(x+y)^n = (x+y)\cdots(x+y) = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n} y^{\mathrm{n}}$$

(1) Coefficients of $x^{n-k}y^k$ and $x^k y^{n-k}$ must be the same $\Longrightarrow$ $\binom{n}{k} = \binom{n}{n-k}$

(2) $(x+y)^{n+1} = (x+y)^n(x+y) \Longrightarrow$

$$\binom{n+1}{0} x^{n+1} + \binom{n+1}{1} x^n y + \cdots + \binom{n+1}{k} x^{n+1-k}y^k + \cdots + \binom{n+1}{n+1} y^{\mathrm{n+1}} =$$

$$\left(\binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n} y^{\mathrm{n}}\right)(x+y) =$$

$$\binom{n}{0} x^n + \left(\binom{n}{1} + \binom{n}{0}\right) x^n y + \cdots + \left(\binom{n}{k} + \binom{n}{k-1}\right) x^{n+1-k}y^k + \cdots + \binom{n}{n} y^{\mathrm{n}}$$

Properties

By comparing the coefficients, we conclude that: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ ♠

Here is another way of proving ♠ :

Suppose you want to choose a subset S of size k from the set $\{ a_1, \ldots, a_{n+1} \}$

- Total number of choices: $\binom{n+1}{k}$
- Number of those S that do not include a_{n+1} : $\binom{n}{k}$
- Number of those S that do include a_{n+1} : $\binom{n}{k-1}$

Pascal's triangle

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\cdots \qquad \vdots \qquad \ddots
\end{array}
$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\cdots \qquad \vdots \qquad \ddots
\end{array}
$$

Exercises:

- Prove the followings:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n\binom{n}{n} = 0$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

What is the formula for $\binom{n}{k}$

- Lemma. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Comment: $n! = 1 \times 2 \times \cdots \times n.$ and $0! = 1$
- Proof of Lemma:

To choose k elements from a set of size n:

You have (1) n choice for the first pick, (2) $n-1$ choice for the second pick … (k) $n-k+1$ choice for the last pick

- Totally: that is $n(n-1)\cdots(n-k+1)$ choices
- However, the order in which the k items are chosen is not important

Proof continued

- That is

$$k! = \text{the number of ways to order } k \text{ things}$$

redundancy.

Therefore: $\binom{n}{k} = \frac{n\times(n-1)\times\cdots\times(n-k+1)}{k!} = \frac{n\times(n-1)\times\cdots\times(n-k+1)\times(n-k)!}{k!\ (n-k)!} = \frac{n!}{k!(n-k)!}$

More variables

What is the expansion formula for more than two variables?

$$(x_1 + x_2 + \cdots + x_k)^n$$

- Example:

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

$$(x + y + z)^3 =$$

$$1\,(x^3 + y^3 + z^3) + 3\,(x^2y + y^2x + x^2z + z^2x + y^2z + z^2y) + 6\,xyz$$

Sigma notation

- In order to write big sums with compact notation, people use the notation

$$\sum_{lower\ limit\ of\ index}^{upper\ limit\ of\ index} summand$$

- Example

Instead of writing $\binom{n}{0}\ x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n} y^{\mathrm{n}}$

We write $\sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k$ or $\sum_{0\le k\le n} \binom{n}{k} x^{n-k}y^k$

General expansion formula

- $(x_1 + x_2 + \cdots + x_k)^n = \sum_{m_1+\cdots+m_k=n} \binom{n}{m_1\ m_2 \cdots m_k} x_1^{m_1} x_2^{m_2} \ldots x_k^{m_k}$
- $\binom{n}{m_1\ m_2 \cdots m_k} =$

  Number of ways to divide a set of n objects into k <u>ordered</u> groups of sizes $m_1, m_2, \ldots, m_k$

- Note that for $k = 2, \binom{n}{m_1\ m_2} = \binom{n}{m_1}$ in our previous notation: Once you pick the first group the rest automatically go the next group.
- $\binom{n}{m_1\ m_2 \cdots m_k} = \frac{n!}{m_1!\ m_2! \cdots m_k!}$ Example: $\binom{6}{2\ 2\ 2} = \frac{6!}{2!2!2!} = \frac{720}{2\times2\times2} = 90$

Stirling numbers nt

Stirling number $S(n,k) =$ the number of ways to partition n different objects into k non-empty groups, where there is no order on the sets

- There is no easy formula for $S(n,k)$ but the can be calculated recursively

Examples:

$$S(n,n) = 1$$

$$S(n,n-1) = \binom{n}{2}$$

Recursive formula

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$

- Proof: Set $= \{1,2,3,\cdots,n\}$
- There are two types of partitions of this set into k non-empty un-ordered subsets
- (1) Those for which $\{n\}$ a group by itself
- (2) Those for which n belongs to a larger subset
- #(1) = $S(n-1,k-1)$: you need to divide $\{1,2,\cdots n-1\}$into $k-1$ subsets
- #(2) = $k \cdot S(n-1,k)$: you divide $\{1,2,\cdots n-1\}$ into k subsets and put n into one of them

Problems*

- You are creating a 4-digit pin code. How many choices are there in the following cases?

(a) With no restriction.

(b) No digit is repeated.

(c) No digit is repeated, digit number 3 is a 0.

(d) No digit is repeated, and they must appear in increasing order.

(e) No digit is repeated, 2 and 5 must be present.

*: Questions taken from Combinatorics and counting, by Per Alexandersson

Problems*

- How many shuffles are there of a deck of cards, such that A♥ is not directly on top of K♥, and A♠ is not directly on top of K♠?

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Problems*

- How many different words can be created by rearranging the letters in SELFIESTICK?

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Problems*

- In how many ways can 8 people form couples of two?

*: Questions taken from Combinatorics and counting, by Per Alexandersson

Problems*

- How many integer solutions does $x_1 + x_2 + \cdots + x_k = n$ have, with $x_i \geq 0$?

*: Questions taken from Combinatorics and counting, by Per Alexandersson

Problems

- Show that

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{0\le k\le n} S(n,k)\, x^k$$

What do you get if you plug in $x = n$?